Lett Math Phys (2012) 100:309–325
DOI 10.1007/s11005-011-0541-6

A Variant of the Mukai Pairing via Deformation Quantization

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Received: 21 April 2011 / Revised: 16 November 2011 / Accepted: 17 November 2011
Published online: 30 November 2011 –

Abstract. Let X be a smooth projective complex variety. The Hochschild homology $\mathrm{HH}_\bullet(X)$ of X is an important invariant of X, which is isomorphic to the Hodge cohomology of X via the Hochschild–Kostant–Rosenberg isomorphism. On $\mathrm{HH}_\bullet(X)$, one has the Mukai pairing constructed by Caldararu. An explicit formula for the Mukai pairing at the level of Hodge cohomology was proven by the author in an earlier work (following ideas of Markarian). This formula implies a similar explicit formula for a closely related variant of the Mukai pairing on $\mathrm{HH}_\bullet(X)$. The latter pairing on $\mathrm{HH}_\bullet(X)$ is intimately linked to the study of Fourier–Mukai transforms of complex projective varieties. We give a new method to prove a formula computing the aforementioned variant of Caldararu's Mukai pairing. Our method is based on some important results in the area of deformation quantization. In particular, we use part of the work of Kashiwara and Schapira on Deformation Quantization modules together with an algebraic index theorem of Bressler, Nest and Tsygan. Our new method explicitly shows that the "Noncommutative Riemann–Roch" implies the classical Riemann–Roch. Further, it is hoped that our method would be useful for generalization to settings involving certain singular varieties.

Mathematics Subject Classification (2000). 19L10, 14C40, 19D55, 53D55.

Keywords. Mukai pairing, Hochschild homology, periodic cyclic homology, algebraic index theorem, Euler class, deformation quantization.

1. Introduction

1.1. BACKGROUND

Let X denote a smooth projective complex variety (we remind the reader that X has the Zariski topology). We denote the corresponding (compact) complex manifold by X^{an}. The Hochschild homology $\mathrm{HH}_\bullet(X)$ is an important algebraic geometric invariant of X. The Hochschild–Kostant–Rosenberg (HKR) isomorphism $I_{\mathrm{HKR}} : \mathrm{HH}_\bullet(X) \to \oplus_i \mathrm{H}^{i-\bullet}(X, \Omega^i_X)$ identifies $\mathrm{HH}_\bullet(X)$ with the Hodge cohomology of X (as $\mathbb{C}$-vector spaces). We also recall that if X and Y are two smooth projective complex varieties, then one has a Kunneth isomorphism identifying $\mathrm{HH}_\bullet(X) \otimes \mathrm{HH}_\bullet(Y)$ with $\mathrm{HH}_\bullet(X \times Y)$ (see [16,23,27]).

One key reason for the importance of Hochschild homology is its connection to Fourier–Mukai transforms in algebraic geometry. Recall that if $\mathcal{E}$ is an element of the bounded derived category $\mathrm{D}^b(X \times Y)$ of coherent sheaves on $X \times Y$, then $\mathcal{E}$ may be viewed as the kernel of a Fourier–Mukai transform

$$\Phi_{\mathcal{E}} : \mathrm{D}^b(X) \to \mathrm{D}^b(Y), \quad \mathcal{F} \mapsto \pi_{Y*}(\mathcal{E} \otimes \pi_X^* \mathcal{F}). \tag{1}$$

A corresponding Fourier–Mukai transform

$$\Phi_{\mathcal{E}}^{\mathrm{HH}} : \mathrm{HH}_\bullet(X) \to \mathrm{HH}_\bullet(Y)$$

is defined on Hochschild homologies in several a priori different ways in [5,15,27]. These definitions have since been shown to be equivalent in [23]. In particular, if $\mathcal{E} \in \mathrm{D}^b(X)$, one obtains

$$\Phi_{\mathcal{E}}^{\mathrm{HH}} : \mathbb{C} \cong \mathrm{HH}_\bullet(\mathrm{pt}) \to \mathrm{HH}_\bullet(X)$$

by viewing $\mathcal{E}$ as the kernel of a Fourier Mukai transform from pt to X. Thus, one can define the *Hochschild class*

$$\mathrm{Ch}^{\mathrm{HH}}(\mathcal{E}) := \Phi_{\mathcal{E}}^{\mathrm{HH}}(1) \in \mathrm{HH}_0(X)$$

of $\mathcal{E}$ as in [5]. The notation $\mathrm{Ch}^{\mathrm{HH}}(\mathcal{E})$ is used to remind the reader of the close analogy and relation of the Hochschild class with the Chern character. Indeed, Theorem 4.5 of [6] shows that

$$I_{\mathrm{HKR}}(\mathrm{Ch}^{\mathrm{HH}}(\mathcal{E})) = \mathrm{Ch}(\mathcal{E}) \in \oplus_i \mathrm{H}^i(X, \Omega_X^i).$$

The Hochschild homology $\mathrm{HH}_\bullet(X)$ has another interesting structure. In [5], Caldararu defined a Mukai pairing $\langle -, - \rangle_M$ on $\mathrm{HH}_\bullet(X)$. On the other hand, one has the Hochschild–Kostant–Rosenberg (HKR) isomorphism $I_{\mathrm{HKR}} : \mathrm{HH}_\bullet(X) \to \oplus_i \mathrm{H}^{i-\bullet}(X, \Omega_X^i)$. The following result was implicitly proven in [18] (and explicitly so in [21] following [18]).

THEOREM 1.

$$\langle a, b \rangle_M = \int_X I_{\mathrm{HKR}}(b) \wedge J(I_{\mathrm{HKR}}(a)) \wedge \mathrm{Td}(T_X).$$

Here, J is the involution multiplying an element of $\mathrm{H}^{i-\bullet}(X, \Omega_X^i)$ by $(-1)^i$. The Mukai pairing is closely related to another pairing $\langle -, - \rangle_{\mathrm{Shk}}$ on $\mathrm{HH}_\bullet(X)$ that was constructed in [23] following Shkyarov in [27]. Different constructions of the same pairing have also appeared in [15,16]. The most transparent definition of $\langle -, - \rangle_{\mathrm{Shk}}$ is the one given in [16]. This definition states that $\langle -, - \rangle_{\mathrm{Shk}}$ is given by the composite map

$$\mathrm{HH}_\bullet(X) \otimes \mathrm{HH}_\bullet(X) \to \mathrm{HH}_\bullet(X \times X) \xrightarrow{\Phi^{\mathrm{HH}}_{\mathcal{O}_\Delta}} \mathrm{HH}_\bullet(\mathrm{pt}) \cong \mathbb{C}.$$

Here, $\mathcal{O}_\Delta := \Delta_* \mathcal{O}_X$ where $\Delta : X \to X \times X$ denotes the diagonal embedding. The object $\mathcal{O}_\Delta$ of $\mathrm{D}^b(X \times X)$ is viewed as the kernel of a Fourier–Mukai transform from $X \times X$ to pt in the above definition.

A careful comparison between $\langle -, - \rangle_{\mathrm{Shk}}$ and $\langle -, - \rangle_M$ was used together with Theorem 1 in [23] to show the following result. The aim of the current paper is to present a new proof of the same result.

THEOREM 2.

$$\langle a, b \rangle_{\mathrm{Shk}} = \int_X I_{\mathrm{HKR}}(a) \wedge I_{\mathrm{HKR}}(b) \wedge \mathrm{Td}(T_X). \tag{2}$$

The usefulness of Theorem 2 stems from the following result about Fourier–Mukai transforms on Hochschild homology (see [16,23,27]).

THEOREM 3. $\Phi_{\mathcal{E}}(x) = \langle x, \mathrm{Ch}^{\mathrm{HH}}(\mathcal{E}) \rangle_{\mathrm{Shk}} \in \mathrm{HH}_\bullet(Y)$ *for any* $x \in \mathrm{HH}_\bullet(X)$.

The right hand side of the formula in Theorem 3 is an abuse of notation. Its correct interpretation is as follows: first, one identifies $\mathrm{HH}_\bullet(X \times Y)$ with $\mathrm{HH}_\bullet(X) \otimes \mathrm{HH}_\bullet(Y)$ via the (inverse of the) Kunneth isomorphism. Hence, $x \otimes \mathrm{Ch}^{\mathrm{HH}}(\mathcal{E})$ is viewed as an element of $\mathrm{HH}_\bullet(X) \otimes \mathrm{HH}_\bullet(X) \otimes \mathrm{HH}_\bullet(Y)$. To this, we apply the homomorphism $\langle -, - \rangle_{\mathrm{Shk}} \otimes \mathrm{id}_{\mathrm{HH}_\bullet(Y)} : \mathrm{HH}_\bullet(X) \otimes \mathrm{HH}_\bullet(X) \otimes \mathrm{HH}_\bullet(Y) \to \mathrm{HH}_\bullet(Y)$ to obtain the R.H.S of the formula in Theorem 3.

Theorem 2 together with Theorem 3 has been of interest in recent years. Not surprisingly, Theorem 1 (equivalently, Theorem 2) implies the Grothendieck Riemann–Roch theorem for smooth projective complex varieties (see [18,19,21]) as well as an explicit version of the Cardy condition (see [24]). Another (possibly more) interesting application of these results is their use for the study of derived equivalences of certain classes of algebraic varieties (for example, K3 surfaces in [11,17]). We remark that as far as (the above cited as well as other) recent applications are concerned, a formula for $\langle -, - \rangle_{\mathrm{Shk}}$ is as useful/suitable as one for $\langle -, - \rangle_M$ (i.e., Theorems 1 and 2 are as useful as one another).

1.2. ABOUT THIS PAPER

In this paper, we provide a different proof of Theorem 2 based on the work of Kashiwara–Schapira [14,15] and an algebraic index theorem of Bressler, Nest and Tsygan in [1–3]. The latter result, one version of which is Theorem 4.6.1 of [2], was proven as part of the authors' resolution of a conjecture of Schapira and Schneiders (Conjecture 8.5 of [26]) pertaining to the Euler class of $\mathcal{D}$-modules. Unlike the earlier approach from [18,21,23] (also see [25] for further details), this approach requires that we work over $\mathbb{C}$. However, it gives a clear connection (hitherto missing) between the computation of a "Mukai pairing" and a large body of work in deformation quantization, algebraic index theorems and related topics. Specifically, our paper shows how the "noncommutative Riemann–Roch" (which is what the Bressler–Nest–Tsygan result amounts to) implies the classical Riemann–Roch.

Our new proof of Theorem 2 proceeds as follows. Results from Section 5 of [15] are first used in Section 2 to reduce Theorem 2 to the statement that the Euler class $\mathrm{Eu}(\mathcal{O}_X)$ of the structure sheaf of X coincides with the Todd genus $\mathrm{Td}(T_X)$ of the tangent bundle of X. Till recently, this statement was a conjecture of Kashiwara dating back to 1991. In what follows, we shall refer to this statement as Kashiwara's conjecture.

We prove Kashiwara's conjecture in Section 3. Our proof uses a proposition (Proposition 5) comparing the Hochschild homology $\mathrm{HH}_\bullet(X) := \mathrm{HH}_\bullet(\mathcal{O}_X)$ with the Hochschild homology $\mathrm{HH}_\bullet(\mathcal{D}_{X^{\mathrm{an}}})$ of the sheaf of holomorphic differential operators on X^{an}. Proposition 5 is proven via some standard arguments from Proposition 3, a similar comparison result for periodic cyclic homologies. Proposition 3 in turn follows from Theorem 4.6.1 of [2]. In order to stick to the main narrative in Section 3, we postpone the proof of Proposition 3 to Section 4.

The argument that Proposition 5 implies Kashiwara's conjecture has three main ingredients. These are: the proof of Proposition 5.2.3 of [15], the Riemann–Roch–Hirzebruch theorem for holomorphic differential operators from [7,22] and Proposition 7. Proposition 7, a technical result computing the Hochschild homology $\mathrm{HH}_\bullet(\mathrm{Perf}(\mathcal{D}_X))$ of the DG-category of perfect right $\mathcal{D}_X$-modules, is analogous to Theorem 5.2 of [12]. Its proof uses generalizations due to Yao (in [32]) of certain deep propositions of Thomason and Trobaugh in [30].

We finish this introduction by pointing out that Kashiwara's conjecture has been proven in [9] using a deformation to the normal cone argument. While the (interesting) approach in [9] is far more concise than the one via [18,21,23], the argument there is geometric and not intrinsic to X. Readers with some background in deformation quantization and algebraic index theory would also find the approach to Theorem 2 in this paper far more concise than the earlier one (that in [18,21, 23]), while remaining algebraic and intrinsic to X in nature. Further, unlike the earlier approach, this method is likely to lend itself to generalization to more general settings involving certain singular varieties.

2. Preliminaries

Let $\omega_X := \Omega^n_X[n]$. Let $\Delta : X \to X \times X$ denote the diagonal embedding. Recall from [15] that one has the following commutative diagram in the bounded derived category $\mathrm{D}^b(\mathcal{O}_X)$ of coherent sheaves on X.

$$\begin{array}{ccc} \Delta^*\Delta_*\mathcal{O}_X & \xrightarrow[\text{td}]{} & \Delta^!\Delta_*\omega_X \\ \big\downarrow I_{\mathrm{HKR}} & & \big\uparrow \widehat{I_{\mathrm{HKR}}} \\ \oplus_i \Omega^i_X[i] & \xrightarrow[\tau]{} & \oplus_i \Omega^i_X[i] \end{array}$$

Here, the map td is as constructed in Section 5.2 of [15] and the map $\widehat{I_{\mathrm{HKR}}}$ is as constructed in [15], Section 5.4.[1] Let D denote the map on hypercohomology induced by $\mathrm{td}\colon \Delta^*\Delta_*\mathcal{O}_X \to \Delta^!\Delta_*\omega_X$. Let $I_{\mathrm{HKR}}, \widehat{I_{\mathrm{HKR}}}$ and τ continue to denote the maps induced on hypercohomology by $I_{\mathrm{HKR}}, \widehat{I_{\mathrm{HKR}}}$ and τ, respectively. Applying hypercohomologies, one obtains the following commutative diagram.

$$\begin{array}{ccc} \mathbb{H}^{-\bullet}(X, \Delta^*\Delta_*\mathcal{O}_X) & \xrightarrow[\mathrm{D}]{} & \mathbb{H}^{-\bullet}(X, \Delta^!\Delta_*\omega_X) \\ \big\downarrow I_{\mathrm{HKR}} & & \big\uparrow \widehat{I_{\mathrm{HKR}}} \\ \oplus_i \mathrm{H}^{i-\bullet}(X, \Omega^i_X) & \xrightarrow[\tau]{} & \oplus_i \mathrm{H}^{i-\bullet}(X, \Omega^i_X) \end{array} \tag{3}$$

Kashiwara and Schapira show us in [15][2,3] that

PROPOSITION 1. *Theorem 2 is equivalent to the assertion that the map τ in (3) is the wedge product with $Td(TX)$.*

Proof. Let X, Y be smooth projective varieties over $\mathbb{C}$. Recall that any $\Phi \in \mathrm{D}^b_{\mathrm{coh}}(X \times Y)$ gives an integral transform $\Phi^{\mathrm{cal}}_*\colon \mathrm{HH}_\bullet(X) \to \mathrm{HH}_\bullet(Y)$ (see Section 4.3 of [5]). On hypercohomologies, Corollary 4.2.2 of [15] yields a pairing

$$\langle -, - \rangle_{\mathrm{KS}}\colon \mathrm{HH}_\bullet(X) \otimes \mathrm{HH}_\bullet(X) \to \mathbb{C}.$$

We remark that $\mathrm{HH}_\bullet(X)$ is also the hypercohomology of the complex of Hochschild chains of $\mathcal{O}_X^{\mathrm{op}}$, which is equal to $\mathrm{HH}_\bullet(X)$ since $\mathcal{O}_X^{\mathrm{op}} = \mathcal{O}_X$. In particular, we are not making this identification via the duality map described at the end of Section 4.1 of [15]. Lemma 4.3.4 of [15] then tells us that after identifying $\mathrm{HH}_\bullet(X \times Y)$ with $\mathrm{HH}_\bullet(Y) \otimes \mathrm{HH}_\bullet(X)$,[4]

$$\Phi^{\mathrm{cal}}_*(\alpha) = \langle \mathrm{Ch}(\Phi), \alpha \rangle_{\mathrm{KS}}. \tag{4}$$

We point out that the right hand side of Equation (4) involves an abuse of notation and that its correct interpretation is analogous to that of the right hand side of Theorem 3. Let $\Phi = \mathcal{O}_\Delta$ (Δ here denoting the diagonal in $X \times X$). In this case,

[1] A similar map has been constructed in Section 1 of [21].

[2] We remark that all constructions/results in Chapter 5 of [15], which are done in the setting of complex manifolds, work in the algebraic setting that we are working in.

[3] As mentioned at the beginning of Chapter 5 of [15], most of the work of Chapter 5 in [15] dates back to a letter [13] written by Kashiwara to Schapira in 1991.

[4] $X \times Y$ is viewed as $Y \times X$ while making this identification.

$\Phi_*^{\mathrm{cal}} = \mathrm{id}$ (see Section 5 of [5]). Then, by Theorem 5 of [23][5], $\mathrm{Ch}(\Phi) = \sum_i e_i \otimes f_i$, where the e_i and f_j are homogenous bases of $\mathrm{HH}_\bullet(X)$ such that $\langle f_j, e_i\rangle_{\mathrm{Shk}} = \delta_{ij}$. On the other hand, Equation (4) applied to $\alpha = e_i$ tells us that $\langle f_j, e_i\rangle_{\mathrm{KS}} = \delta_{ij}$, thus showing that $\langle -,-\rangle_{\mathrm{KS}} = \langle -,-\rangle_{\mathrm{Shk}}$. Finally, the end of Section 5.4 of [15] shows us that

$$\langle a, b\rangle_{\mathrm{KS}} = \int_X I_{\mathrm{HKR}}(a) \wedge \tau(I_{\mathrm{HKR}}(b)).$$

□

We therefore, need to show that $\tau = (- \wedge \mathrm{Td}(TX))$. In our method, the following proposition from [15], Chapter 5 is the first step in this direction.

PROPOSITION 2. (i) *$\Delta^*\Delta_*\mathcal{O}_X$ is a ring object in $D^b(\mathcal{O}_X)$, and $\Delta^!\Delta_*\omega_X$ is a left module object over $\Delta^*\Delta_*\mathcal{O}_X$ in $D^b(\mathcal{O}_X)$.*
(ii) *Further, td is a morphism of left $\Delta^*\Delta_*\mathcal{O}_X$ modules in $D^b(\mathcal{O}_X)$.*

Proof. The ring structure of $\Delta^*\Delta_*\mathcal{O}_X$ in $\mathrm{D}^b(\mathcal{O}_X)$ is given by the composite map

$$\Delta^*\Delta_*\mathcal{O}_X \otimes^{\mathbb{L}}_{\mathcal{O}_X} \Delta^*\Delta_*\mathcal{O}_X \cong \Delta^*(\Delta_*\mathcal{O}_X \otimes^{\mathbb{L}}_{\mathcal{O}_{X\times X}} \Delta_*\mathcal{O}_X) \xrightarrow{\Delta^*\mu} \Delta^*\Delta_*\mathcal{O}_X,$$

where μ is induced by the product map $\Delta_*\mathcal{O}_X \otimes_{\mathcal{O}_{X\times X}} \Delta_*\mathcal{O}_X$.

The module structure of $\Delta^!\Delta_*\omega_X$ over $\Delta^*\Delta_*\mathcal{O}_X$ is realized via the composite map

$$\Delta^*\Delta_*\mathcal{O}_X \otimes^{\mathbb{L}}_{\mathcal{O}_X} \Delta^!\Delta_*\omega_X \cong \Delta^!(\Delta_*\mathcal{O}_X \otimes^{\mathbb{L}}_{\mathcal{O}_{X\times X}} \Delta_*\omega_X) \xrightarrow{\Delta^!\nu} \Delta^!\Delta_*\omega_X.$$

Here, ν is the composite map

$$\Delta_*\mathcal{O}_X \otimes^{\mathbb{L}}_{\mathcal{O}_{X\times X}} \Delta_*\omega_X \cong \Delta_*(\Delta^*\Delta_*\mathcal{O}_X \otimes_{\mathcal{O}_X} \omega_X) \to \Delta_*\omega_X$$

the last arrow being induced by the adjunction $\Delta^*\Delta_*\mathcal{O}_X \to \mathcal{O}_X$.

The morphism td was constructed in [15] as follows.

$$\begin{aligned}\Delta^*\Delta_*\mathcal{O}_X &\cong \mathcal{O}_X \otimes^{\mathbb{L}}_{\mathcal{O}_X} \Delta^*\Delta_*\mathcal{O}_X \\ &\cong \Delta^!(\mathcal{O}_X \boxtimes \omega_X) \otimes^{\mathbb{L}}_{\mathcal{O}_X} \Delta^*\Delta_*\mathcal{O}_X \\ &\cong \Delta^!((\mathcal{O}_X \boxtimes \omega_X) \otimes^{\mathbb{L}}_{\mathcal{O}_{X\times X}} \Delta_*\mathcal{O}_X)\end{aligned}$$

$$\Delta^!((\mathcal{O}_X \boxtimes \omega_X) \otimes^{\mathbb{L}}_{\mathcal{O}_{X\times X}} \Delta_*\mathcal{O}_X) \cong \Delta^!\Delta_*\omega_X$$

That td is a morphism of left $\Delta^*\Delta_*\mathcal{O}_X$-modules is more or less a direct consequence of the fact that $\otimes^{\mathbb{L}}_{\mathcal{O}_X}$ is associative. □

[5]Note that we are not using any part of [23] that depends on the Mukai pairing formula computed in [18,21].

COROLLARY 1. *For all* $\alpha \in \oplus_i H^{i-\bullet}(X, \Omega^i_X)$, $\tau(\alpha) = \alpha \wedge \tau(1)$.

Proof. The ring structure of $\Delta^*\Delta_*\mathcal{O}_X$ induces a product $\bullet$ on $\mathbb{H}^{-\bullet}(X, \Delta^*\Delta_*\mathcal{O}_X)$. By Proposition 2,

$$\mathrm{D}(a \bullet b) = a \bullet \mathrm{D}(b)$$

for all $a, b \in \mathbb{H}^{-\bullet}(X, \Delta^*\Delta_*\mathcal{O}_X)$. It follows from Lemma 5.4.7 of [15] that for all $a, b \in \mathbb{H}^{-\bullet}(X, \Delta^*\Delta_*\mathcal{O}_X)$,

$$\widehat{I_{\mathrm{HKR}}}(I_{\mathrm{HKR}}(a) \wedge \beta) = a \bullet \widehat{I_{\mathrm{HKR}}}(\beta).$$

The desired corollary now follows from the fact that I_{HKR} and $\widehat{I_{\mathrm{HKR}}}$ are isomorphisms. □

Recall that for any $E \in \mathrm{D}^b(\mathcal{O}_X)$, one has the *Chern character* $\mathrm{ch}(E) \in \mathbb{H}^0(X, \Delta^*\Delta_*\mathcal{O}_X)$. By Theorem 4.5 of [6], $I_{\mathrm{HKR}}(\mathrm{ch}(E))$ is the Chern character of E in the classical sense. The *Euler class* $\mathrm{Eu}(E)$ is defined as the element $\widehat{I_{\mathrm{HKR}}}^{-1}(\mathrm{D}(\mathrm{ch}((E)))$ of $\oplus_i \mathrm{H}^i(X, \Omega^i_X)$. Note $\tau(1) = \mathrm{Eu}(\mathcal{O}_X)$. In order to compute the $\langle -, - \rangle_{\mathrm{Shk}}$, we therefore, need to show that

$$\mathrm{Eu}(\mathcal{O}_X) = \mathrm{Td}(T_X).$$

Before we proceed, let us make a clarification. Recall that $\Delta^*\Delta_*\mathcal{O}_X$ is represented in the derived category $\mathrm{D}^-(\mathcal{O}_X)$ of bounded above complexes of quasi-coherent sheaves on X by the complex of $\widehat{\mathcal{C}_\bullet}(\mathcal{O}_X)$ of completed Hochschild chains (after turning it into a cochain complex by inverting degrees). Recall from [33] that $\widehat{\mathcal{C}_n}(\mathcal{O}_X) := \varprojlim_k \frac{\mathcal{O}_X^{\otimes n+1}}{I_n^k}$, where I_n is the kernel of the product map $\mathcal{O}_X^{\otimes n+1} \to \mathcal{O}_X$. Let $\mathcal{C}_\bullet(\mathcal{O}_X)$ be the complex of sheaves of X associated with the complex of presheaves $U \mapsto \mathrm{C}_\bullet(\Gamma(U, \mathcal{O}_X))$ (the Hochschild chain complex here being the naive algebraic one). One similarly defines $\mathcal{C}^{\mathrm{red}}_\bullet(\mathcal{O}_X)$ using reduced Hochschild chains. There are natural maps $\mathcal{C}^{\mathrm{red}}_\bullet(\mathcal{O}_X) \leftarrow \mathcal{C}_\bullet(\mathcal{O}_X) \to \widehat{\mathcal{C}_\bullet}(\mathcal{O}_X)$ of complexes of sheaves on X which are quasi-isomorphisms. In the following section, when thinking of the complex of Hochschild chains on X, we shall be thinking of $\mathcal{C}^{\mathrm{red}}_\bullet(\mathcal{O}_X)$ (which has the same hypercohomology as $\widehat{\mathcal{C}_\bullet}(\mathcal{O}_X)$).

3. The Euler Class of $\mathcal{O}_X$

It remains to show that $\mathrm{Eu}(\mathcal{O}_X) = \mathrm{Td}(T_X)$. This fact was originally conjectured (in 1991) by Kashiwara in [13]. The original intrinsic computation proving this from [18] (see [21] for details) is very lengthy and involved. Further, its connections to deformation quantization and related areas are not clear. Another, more recent proof due to [9] uses deformation to the normal cone. We now sketch our new approach to this question. Let $\mathcal{D}_X$ denote the sheaf of (algebraic) differential operators on X. Recall that the Hochschild–Kostant–Rosenberg quasi-isomorphism on Hochschild chains induces an isomorphism $I_{\text{HKR}}: \text{HC}_0^{\text{per}}(\mathcal{O}_X) \to \prod_{p=-\infty}^{\infty} \text{H}^{2p}(X^{\text{an}}, \mathbb{C})$. On the other hand, a construction very similar to the trace density construction of Engeli–Felder on Hochschild chains induces an isomorphism $\chi: \text{HC}_0^{\text{per}}(\mathcal{D}_{X^{\text{an}}}) \to \prod_{p=-\infty}^{\infty} \text{H}^{2n-2p}(X^{\text{an}}, \mathbb{C})$ (see [7,20,28]). Further, one has a natural map $(-)^{\text{an}}: \text{HC}_0^{\text{per}}(\mathcal{D}_X) \to \text{HC}_0^{\text{per}}(\mathcal{D}_{X^{\text{an}}})$[6]. The natural homomorphism $\mathcal{O}_X \to \mathcal{D}_X$ of sheaves of algebras on X induces maps on Hochschild as well as negative cyclic and periodic cyclic homologies. These maps shall be denoted by ι. The following proposition is closely related to a Theorem in [1] (also see [2,3]). It will be proved in Section 4.

PROPOSITION 3. *The following diagram commutes:*

$$\begin{array}{ccc}
\text{HC}_0^{\text{per}}(\mathcal{O}_X) & \xrightarrow[(-)^{\text{an}}\circ\iota]{} & \text{HC}_0^{\text{per}}(\mathcal{D}_{X^{\text{an}}}) \\
\downarrow{\scriptstyle I_{\text{HKR}}} & & {\scriptstyle\chi}\downarrow \\
\prod_{p=-\infty}^{\infty} \text{H}^{2p}(X^{\text{an}}, \mathbb{C}) & \xrightarrow{(-\wedge Td(T_X))} & \prod_{p=-\infty}^{\infty} \text{H}^{2n-2p}(X^{\text{an}}, \mathbb{C})
\end{array}$$

Note that for any sheaf of algebras $\mathcal{A}$ on X, one has natural maps $\text{HC}_0^{-}(\mathcal{A}) \to \text{HC}_0^{\text{per}}(\mathcal{A})$ and $\text{HC}_0^{-}(\mathcal{A}) \to \text{HH}_0(\mathcal{A})$. Also recall that one has a natural projection $\text{H}^{2p}(X^{\text{an}}, \mathbb{C}) \to \text{H}^{p,p}(X^{\text{an}}, \mathbb{C})$ for all p.

PROPOSITION 4. *The following diagrams commute:*

(a)

$$\begin{array}{ccc}
\text{HC}_0^{-}(\mathcal{O}_X) & \xrightarrow{I_{\text{HKR}}} & \prod_{p=-\infty}^{\infty} \text{H}^{2p}(X^{\text{an}}, \mathbb{C}) \\
\downarrow & & \downarrow \\
\text{HH}_0(\mathcal{O}_X) & \xrightarrow{I_{\text{HKR}}} & \oplus_p \text{H}^{p,p}(X^{\text{an}}, \mathbb{C})
\end{array}$$

(b)

$$\begin{array}{ccc}
\text{HC}_0^{-}(\mathcal{D}_{X^{\text{an}}}) & \xrightarrow{\chi} & \prod_{p=-\infty}^{\infty} \text{H}^{2n-2p}(X^{\text{an}}, \mathbb{C}) \\
\downarrow & & \downarrow \\
\text{HH}_0(\mathcal{D}_{X^{\text{an}}}) & \xrightarrow{\chi} & \text{H}^{2n}(X^{\text{an}}, \mathbb{C})
\end{array}$$

[6] Indeed, if $f: X^{\text{an}} \to X$ is the canonical map, one has a natural map $f^{-1}(\mathcal{CC}_\bullet^{\text{per}}(\mathcal{D}_X)) \to \mathcal{CC}_\bullet^{\text{per}}(\mathcal{D}_{X^{\text{an}}})$ of complexes of sheaves on X^{an}, and hence in the derived category $\text{D}(\text{Sh}_{\mathbb{C}}(X^{\text{an}}))$ of sheaves of $\mathbb{C}$-vector spaces on X^{an}. By adjunction, one gets a natural map $\mathcal{CC}_\bullet^{\text{per}}(\mathcal{D}_X) \to Rf_*(\mathcal{CC}_\bullet^{\text{per}}(\mathcal{D}_{X^{\text{an}}}))$, to which we apply $R\Gamma(X, -)$. Rf_* and $R\Gamma$ extend to $\text{D}(\text{Sh}_{\mathbb{C}}(X^{\text{an}}))$ and $\text{D}(\text{Sh}_{\mathbb{C}}(X))$, respectively, since f_* and $\Gamma(X, -)$ have finite cohomological dimension.

(c)

$$\begin{array}{ccc} \mathrm{HC}_0^-(\mathcal{O}_X) & \xrightarrow{(-)^{\mathrm{an}}\circ\iota} & \mathrm{HC}_0^-(\mathcal{D}_{X^{\mathrm{an}}}) \\ \downarrow & & \downarrow \\ \mathrm{HH}_0(\mathcal{O}_X) & \xrightarrow{(-)^{\mathrm{an}}\circ\iota} & \mathrm{HH}_0(\mathcal{D}_{X^{\mathrm{an}}}) \end{array}$$

Proof. We prove part (a), leaving the proof of the remaining parts to the reader. Let C^p denote the complex

$$0 \to \Omega_X^p \xrightarrow{d_{\mathrm{DR}}} \Omega_X^{p+1} \xrightarrow{d_{\mathrm{DR}}} \cdots$$

of sheaves (of $\mathbb{C}$-vector spaces) on X with Ω_X^p in (cohomological) degree 0. There is a natural map $C^p \to \Omega_X^p$ of complexes of sheaves on X given by the identity on Ω_X^p (here, Ω_X^p is thought of as a complex concentrated in degree 0). One also has a natural map of complexes $C^p \to \Omega^\bullet_{X,DR}[p]$ of sheaves on X, where $\Omega^\bullet_{X,DR}[p]$ denotes the algebraic De-Rham complex of X with a shift. The following diagram clearly commutes:

$$\begin{array}{ccc} \mathrm{HC}_0^-(\mathcal{O}_X) & \xrightarrow{I_{\mathrm{HKR}}} & \mathbb{H}^0(X, \oplus_p C^p[p]) \\ \downarrow & & \downarrow \\ \mathrm{HH}_0(\mathcal{O}_X) & \xrightarrow{I_{\mathrm{HKR}}} & \mathbb{H}^0(X, \oplus_p \Omega_X^p[p]) \end{array}$$

It therefore, suffices to show that the natural map $\mathbb{H}^0(X, C^p[p]) \to \mathrm{H}^p(X, \Omega_X^p)$ coincides with the composite map[7] $\mathbb{H}^0(X, C^p[p]) \to \mathbb{H}^0(X, \Omega^\bullet_{X,DR}[2p]) \cong \mathrm{H}^{2p}(X^{\mathrm{an}}, \mathbb{C}) \to \mathrm{H}^{p,p}(X^{\mathrm{an}}, \mathbb{C})$ after one identifies $\mathrm{H}^p(X, \Omega_X^p)$ with $\mathrm{H}^{p,p}(X^{\mathrm{an}}, \mathbb{C})$. Note that the hypercohomology $\mathbb{H}^0(X, C^p[p])$ may be computed by passing to X^{an} and replacing each Ω_X^i by the corresponding Dolbeault resolution to obtain a double complex, the 0th cohomology of whose total complex is $\mathbb{H}^0(X, C^p[p])$. Hence, any class in $\alpha \in \mathbb{H}^0(X, C^p[p])$ is represented by a harmonic $2p$-form $\omega := \oplus_{p \leqslant r \leqslant 2p} \omega_{r,2p-r}$ on X^{an}, where $\omega_{r,2p-r}$ is a harmonic $(r, 2p-r)$-form on X^{an}. The image of α in $\mathrm{H}^{2p}(X^{\mathrm{an}}, \mathbb{C})$ is also represented by ω. Clearly, the image of α in $\mathrm{H}^p(X, \Omega_X^p)$ is represented by $\omega_{p,p}$, which coincides with the projection from $\mathrm{H}^{2p}(X^{\mathrm{an}}, \mathbb{C})$ to $\mathrm{H}^{p,p}(X^{\mathrm{an}}, \mathbb{C})$ applied to the class of ω in $\mathrm{H}^{2p}(X^{\mathrm{an}}, \mathbb{C})$. □

[7] Indeed, the composition of direct sum (over p) of the composite maps $\mathbb{H}^0(X, C^p[p]) \to \mathbb{H}^0(X, \Omega^\bullet_{X,DR}[2p]) \cong \mathrm{H}^{2p}(X^{\mathrm{an}}, \mathbb{C})$ with $I_{\mathrm{HKR}} : \mathrm{HC}_0^-(\mathcal{O}_X) \to \mathbb{H}^0(X, \oplus_p C^p[p])$ is what we denote by I_{HKR} in the statement of this proposition.

PROPOSITION 5. *The following diagram commutes:*

$$\begin{array}{ccc} \mathrm{HH}_0(\mathcal{O}_X) & \xrightarrow{(-)^{\mathrm{an}}\circ\iota} & \mathrm{HH}_0(\mathcal{D}_{X^{\mathrm{an}}}) \\ \downarrow{\scriptstyle I_{\mathrm{HKR}}} & & {\scriptstyle \chi}\downarrow \\ \oplus_p \mathrm{H}^{p,p}(X^{\mathrm{an}},\mathbb{C}) & \xrightarrow[(-\wedge Td(T_X))_{2n}]{} & \mathrm{H}^{2n}(X^{\mathrm{an}},\mathbb{C}) \end{array}$$

Proof. We note that the natural map $\mathrm{HC}_0^-(\mathcal{O}_X)\to \mathrm{HH}_0(\mathcal{O}_X)$ is surjective. Indeed, after applying I_{HKR}, we are reduced to verifying that $\mathrm{H}^p(X,\mathrm{Ker}(d:\Omega_X^p\to\Omega_X^{p+1}))\to \mathrm{H}^p(X,\Omega_X^p)$ is surjective. By Serre's GAGA, it suffices to verify that $\mathrm{H}^p(X^{\mathrm{an}},\mathrm{Ker}(d:\Omega^p_{X^{\mathrm{an}}}\to\Omega^{p+1}_{X^{\mathrm{an}}}))\to \mathrm{H}^p(X^{\mathrm{an}},\Omega^p_{X^{\mathrm{an}}})$ is surjective. This follows from the fact that any closed (p,p)-form defines an element of $\mathrm{H}^p(X^{\mathrm{an}},\mathrm{Ker}(d:\Omega^p_{X^{\mathrm{an}}}\to\Omega^{p+1}_{X^{\mathrm{an}}}))$ as well.

Hence, any $y\in \mathrm{HH}_0(\mathcal{O}_X)$ lifts to an element $\tilde{y}\in \mathrm{HC}_0^-(\mathcal{O}_X)$. For notational brevity, we denote $\chi\circ(-)^{\mathrm{an}}$ by χ for the rest of this proof. Now, $\chi\circ\iota(y)=(\chi\circ\iota(\tilde{y}))_{2n}$ by Proposition 4, parts (b) and (c). Further, $(\chi\circ\iota(\tilde{y}))_{2n}=(I_{\mathrm{HKR}}(\tilde{y})\wedge \mathrm{Td}(T_X))_{2n}$ by Proposition 3. Finally, $(I_{\mathrm{HKR}}(\tilde{y})\wedge \mathrm{Td}(T_X))_{2n}=(I_{\mathrm{HKR}}(y)\wedge \mathrm{Td}(T_X))_{2n}$ by Proposition 3, part (a) and the fact that $\mathrm{Td}(T_X)\in\oplus_p \mathrm{H}^{p,p}(X^{\mathrm{an}},\mathbb{C})$. $\square$

The following proposition is a crucial point in this note.

PROPOSITION 6. *The following diagram commutes:*

$$\begin{array}{ccc} \mathrm{HH}_0(\mathcal{O}_X) & \xrightarrow{D} & \mathbb{H}^0(X,\Delta^!\Delta_*\omega_X) \\ \downarrow{\scriptstyle (-)^{\mathrm{an}}\circ\iota} & & \downarrow{\scriptstyle (\widehat{I_{\mathrm{HKR}}}^{-1}(-))_{2n}} \\ \mathrm{HH}_0(\mathcal{D}_{X^{\mathrm{an}}}) & \xrightarrow{\chi} & \mathrm{H}^{2n}(X^{\mathrm{an}},\mathbb{C}) \end{array}$$

Proof. Let $\pi:X\to pt$ be the natural projection. The object $\mathcal{O}_X$ of $\mathrm{Perf}(\mathcal{O}_{X\times pt})$ induces a morphism $\pi_*:\mathrm{Perf}(\mathcal{O}_X)\to\mathrm{Perf}(pt)$ in the homotopy category Ho(dg-cat) of DG-categories modulo quasi-equivalences (see Section 8 of [29]). The notation π_* is justified by the fact that the functor from D(Perf(X)) to D(Perf(pt)) induced by π_* is indeed the derived pushforward π_*. This induces a map $\pi_*: \mathrm{HH}_0(\mathcal{O}_X)\to \mathrm{HH}_0(\mathcal{O}_{pt})=\mathbb{C}$ which coincides with the pushforward on Hochschild homologies from [15] (see Theorem 5 of [23]). On the other hand, one has $\pi_*: \oplus_p\mathrm{H}^{p,p}(X^{\mathrm{an}},\mathbb{C})\to \mathrm{H}^0(pt,\mathbb{C})=\mathbb{C}$, which coincides with $\int_{X^{\mathrm{an}}}$. By the proof of Proposition 5.2.3 of [15], $\widehat{I_{\mathrm{HKR}}}^{-1}\circ D$ commutes with π_*. On the other hand, let $\mathrm{Perf}(\mathcal{D}_X)$ denote the DG-category of perfect complexes of (right) $\mathcal{D}_X$-modules that are quasi-coherent as $\mathcal{O}_X$-modules. One has a map $\pi_*^{\mathcal{D}}:\mathrm{Perf}(\mathcal{D}_X)\to\mathrm{Perf}(pt)$ in Ho(dg-cat). The functor induced by $\pi_*^{\mathcal{D}}$ on derived categories maps $M\in \mathrm{D}(\mathrm{Perf}(\mathcal{D}_X))$ to $\pi_*(M^{\mathrm{an}}\otimes^{\mathbb{L}}_{\mathcal{D}_{X^{\mathrm{an}}}}\mathcal{O}_{X^{\mathrm{an}}})$.[8] By Section 8 of [29], $\pi_*^{\mathcal{D}}$ induces a map $\pi_*^{\mathcal{D}}$: HH_0

[8]The latter is indeed in D(Perf(pt)): see [26] for instance.

$(\mathrm{Perf}(\mathcal{D}_X)) \to \mathrm{HH}_0(pt) \cong \mathbb{C}$ on Hochschild homologies. By Proposition 7 at the end of this section, the composite map

$$\mathrm{HH}_\bullet(\mathrm{Perf}(\mathcal{D}_X)) \to \mathrm{HH}_\bullet(\mathcal{D}_X) \xrightarrow{(-)^{\mathrm{an}}} \mathrm{HH}_\bullet(\mathcal{D}_{X^{\mathrm{an}}}) \tag{5}$$

is an isomorphism (the first map in the above composition is the trace map from Section 4 of [12]). $\pi_*^{\mathcal{D}}$ therefore, induces a $\mathbb{C}$-linear functional on $\mathrm{HH}_0(\mathcal{D}_{X^{\mathrm{an}}})$, which we shall continue to denote by $\pi_*^{\mathcal{D}}$. It follows from Theorem 1.1 of [7] and Corollary 1 of [22] that

$$\pi_*^{\mathcal{D}} = \int_{X^{\mathrm{an}}} \circ \chi : \mathrm{HH}_0(\mathcal{D}_{X^{\mathrm{an}}}) \to \mathbb{C}.$$

Since $\int_{X^{\mathrm{an}}} : \mathrm{H}^{2n}(X^{\mathrm{an}}, \mathbb{C}) \to \mathbb{C}$ is an isomorphism, the required proposition follows once we check that $\pi_* = \pi_*^{\mathcal{D}} \circ (-)^{\mathrm{an}} \circ \iota$. This follows from the fact that the diagram

$$\begin{tikzcd}
\mathrm{HH}_0(\mathcal{O}_X) \arrow[r, "(-)^{\mathrm{an}} \circ \iota"] & \mathrm{HH}_0(\mathcal{D}_{X^{\mathrm{an}}}) \\
\mathrm{HH}_0(\mathrm{Perf}(\mathcal{O}_X)) \arrow[u] \arrow[r, "(-) \otimes_{\mathcal{O}_X} \mathcal{D}_X"] & \mathrm{HH}_0(\mathrm{Perf}(\mathcal{D}_X)) \arrow[u]
\end{tikzcd}$$

(the left vertical arrow being the trace isomorphism from Section 4 of [12] and the right vertical arrow being the composite map (5)) commutes as well as the observation that for $E \in \mathrm{D}(\mathrm{Perf}(\mathcal{O}_X))$,

$$\pi_*^{\mathcal{D}} \iota(E) = \pi_*((E \otimes_{\mathcal{O}_X} \mathcal{D}_X)^{\mathrm{an}} \otimes^{\mathbb{L}}_{\mathcal{D}_{X^{\mathrm{an}}}} \mathcal{O}_{X^{\mathrm{an}}}) = \pi_* E^{\mathrm{an}}$$

(recall that $\pi_* E = \pi_* E^{\mathrm{an}}$ in $\mathrm{D}(\mathrm{Perf}(pt))$ by Serre's GAGA). $\square$

Let $\alpha \in \mathrm{HH}_0(\mathcal{O}_X)$ be arbitrary. By Proposition 6,

$$\widehat{I_{\mathrm{HKR}}}^{-1}(\mathrm{D}(\alpha))_{2n} = \chi(\iota(\alpha)^{\mathrm{an}}).$$

By Proposition 5,

$$\chi(\iota(\alpha)^{\mathrm{an}}) = (I_{\mathrm{HKR}}(\alpha) \wedge \mathrm{Td}(T_X))_{2n}.$$

By Corollary 1, $\widehat{I_{\mathrm{HKR}}}^{-1}(\mathrm{D}(\alpha)) = I_{\mathrm{HKR}}(\alpha) \wedge \mathrm{Eu}(\mathcal{O}_X)$. Hence,

$$(I_{\mathrm{HKR}}(\alpha) \wedge \mathrm{Eu}(\mathcal{O}_X))_{2n} = (I_{\mathrm{HKR}}(\alpha) \wedge \mathrm{Td}(T_X))_{2n}$$

for all $\alpha \in \mathrm{HH}_0(\mathcal{O}_X)$. Because wedge-and-integrate is a perfect pairing, $\mathrm{Eu}(\mathcal{O}_X) = \tau(1) = \mathrm{Td}(T_X)$. To complete the proof of Proposition 6, we sketch the proof of the following proposition.

PROPOSITION 7. *$HH_\bullet(Perf(\mathcal{D}_X)) \cong HH_\bullet(\mathcal{D}_{X^{\mathrm{an}}})$. This isomorphism is realized by the composite map* (5).

Proof. One has to verify that the arguments of Keller in Section 5 of [12] go through when $\mathcal{O}_X$ is replaced by $\mathcal{D}_X$. The crucial part here is the analog of Theorem 5.5 of [12] (originally proven as Propositions 5.2.2–5.2.4 of [30]) when $\mathcal{O}_X$ is replaced by $\mathcal{D}_X$. This is done in Propositions 3.3.1–3.3.3 of [32] (which prove the analog of Theorem 5.5 of [12] in a much more general setting: in particular, when $\mathcal{O}_X$ is replaced by $\mathcal{R}_X$, where $\mathcal{R}_X$ is a sheaf of quasi-coherent $\mathcal{O}_X$-algebras (possibly noncommutative)). Let Y be any quasi-compact, quasi-separated scheme over $\mathbb{C}$ with V,W quasi-compact open subschemes of Y such that $Y=V\cup W$. Following the arguments of Sections 5.6 and 5.7 of [12], one obtains a morphism of Mayer–Vietoris sequences

$$\begin{array}{ccccccc}
\mathrm{HH}_i(\mathrm{Perf}(\mathcal{D}_Y)) & \longrightarrow & \mathrm{HH}_i(\mathrm{Perf}(\mathcal{D}_V))\oplus \mathrm{HH}_i(\mathrm{Perf}(\mathcal{D}_W)) & \longrightarrow & \mathrm{HH}_i(\mathrm{Perf}(\mathcal{D}_{V\cap W})) & \longrightarrow & \mathrm{HH}_{i-1}(\mathrm{Perf}(\mathcal{D}_Y)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{HH}_i(\mathcal{D}_{Y^{\mathrm{an}}}) & \longrightarrow & \mathrm{HH}_i(\mathcal{D}_{V^{\mathrm{an}}})\oplus \mathrm{HH}_i(\mathcal{D}_{W^{\mathrm{an}}}) & \longrightarrow & \mathrm{HH}_i(\mathcal{D}_{(V\cap W)^{\mathrm{an}}}) & \longrightarrow & \mathrm{HH}_{i-1}(\mathcal{D}_{Y^{\mathrm{an}}})
\end{array}$$

(for each $i\in\mathbb{Z}$). The vertical arrows in the above diagram are induced by the composite map (5). As in Section 5.9 of [12], we may then reduce the proof of the desired proposition to proving the desired proposition when X is affine with trivial tangent bundle. For the rest of this proof, we assume that this is indeed the case.

Let $\mathcal{D}_X$-mod denote the Abelian category of (right) $\mathcal{D}_X$-modules are quasi-coherent $\mathcal{O}_X$-modules. There is an equivalence of abelian categories between D_X-mod and $\mathcal{D}_X$-mod, where $D_X:=\Gamma(X,\mathcal{D}_X)$ (see [32], example 1.1.5). Hence, one has an equivalence of DG-categories between $\mathrm{Perf}(D_X)$ and $\mathrm{Perf}(\mathcal{D}_X)$ (this follows, for instance, from Lemma 2.2.1 of [32]). This equivalence induces an isomorphism $\mathrm{HH}_\bullet(\mathrm{Perf}(D_X))\stackrel{\cong}{\to}\mathrm{HH}_\bullet(\mathrm{Perf}(\mathcal{D}_X))$. Further, there is a natural map $\mathrm{HH}_\bullet(D_X)\to\mathrm{HH}_\bullet(\mathcal{D}_X)$ such that the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{HH}_\bullet(\mathrm{Perf}(D_X)) & \xrightarrow{\cong} & \mathrm{HH}_\bullet(\mathrm{Perf}(\mathcal{D}_X)) \\
\downarrow{\cong} & & \downarrow \\
\mathrm{HH}_\bullet(D_X) & \longrightarrow & \mathrm{HH}_\bullet(\mathcal{D}_X)
\end{array}$$

In the above diagram, the vertical arrows are trace maps from Section 4 of [12]. For honest algebras, they yield isomorphisms. We are therefore, reduced to verifying that the composite map

$$\mathrm{HH}_\bullet(D_X)\to\mathrm{HH}_\bullet(\mathcal{D}_X)\stackrel{(-)^{\mathrm{an}}}{\to}\mathrm{HH}_\bullet(\mathcal{D}_{X^{\mathrm{an}}}) \tag{6}$$

is an isomorphism. Let $\mathcal{D}^\bullet_{X^{\mathrm{an}}}$ denote the Dolbeault resolution of the sheaf $\mathcal{D}_{X^{\mathrm{an}}}$. This is a sheaf of DG-algebras on X. Let $\mathrm{C}_\bullet(\mathcal{D}^\bullet_{X^{\mathrm{an}}})$ denote the complex of global sections of the complex of completed Hochschild chains on X (see [22], Section 3.3). There is a natural map of complexes $\mathrm{C}_\bullet(D_X)\to\mathrm{C}_\bullet(\mathcal{D}^\bullet_{X^{\mathrm{an}}})$ inducing (6) on homology. To prove that this is a quasi-isomorphism, we filter algebraic and holomorphic differential operators by order and consider the induced map on the E^2-terms of the spectral sequences from Section 3.3 of [4]. This turns out to be induced on homology by the natural map from the algebraic De-Rham complex $(\Omega^{2n-\bullet}(T^*X), d_{\mathrm{DR}}^{\mathrm{alg}})$ to the Dolbeault complex $(\Gamma(X^{\mathrm{an}}, \Omega^{2n-\bullet}_{T^*X^{\mathrm{an}}} \otimes_{\mathcal{O}_{X^{\mathrm{an}}}} \Omega^{0,\bullet}_{X^{\mathrm{an}}}), d+\bar{\partial})$.[9] That this is a quasi-isomorphism amounts to the assertion that the natural map from the algebraic De-Rham complex of X to the smooth De-Rham complex of X^{an} is a quasi-isomorphism (see [10]). □

4. A Proof of Proposition 3

One notes that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{HC}_0^{\mathrm{per}}(\mathcal{O}_X) & \xrightarrow{(-)^{\mathrm{an}}} & \mathrm{HC}_0^{\mathrm{per}}(\mathcal{O}_{X^{\mathrm{an}}}) \\ \downarrow{\scriptstyle\iota} & & {\scriptstyle\iota}\downarrow \\ \mathrm{HC}_0^{\mathrm{per}}(\mathcal{D}_X) & \xrightarrow{(-)^{\mathrm{an}}} & \mathrm{HC}_0^{\mathrm{per}}(\mathcal{D}_{X^{\mathrm{an}}}) \end{array}$$

To prove Proposition 3, it therefore, suffices to show that the following diagram commutes (where $Y := X^{\mathrm{an}}$):

$$\begin{array}{ccc} \mathrm{HC}_0^{\mathrm{per}}(\mathcal{O}_Y) & \xrightarrow[\iota]{} & \mathrm{HC}_0^{\mathrm{per}}(\mathcal{D}_Y) \\ \downarrow{\scriptstyle I_{\mathrm{HKR}}} & & {\scriptstyle\chi}\downarrow \\ \prod_{p=-\infty}^{\infty} \mathrm{H}^{2p}(Y,\mathbb{C}) & \xrightarrow{(-\wedge \mathrm{Td}(T_Y))} & \prod_{p=-\infty}^{\infty} \mathrm{H}^{2n-2p}(Y,\mathbb{C}) \end{array} \tag{7}$$

In other words, we now work with a complex manifold rather than an algebraic variety. Recall that there is a deformation quantization $\mathbb{A}^{\hbar}_{T^*Y}$ of $\mathcal{O}_{T^*Y}[[\hbar]]$ such that $\pi^{-1}\mathcal{D}_Y \hookrightarrow \mathbb{A}^{\hbar}_{T^*Y}[\hbar^{-1}]$ and $\mathbb{A}^{\hbar}_{T^*Y}[\hbar^{-1}]$ are flat over $\pi^{-1}\mathcal{D}_Y$. Here, $\pi : T^*Y \to Y$ is the canonical projection.

In this situation, one has a natural map $\pi^{-1} : \mathrm{HC}_0^{\mathrm{per}}(\mathcal{D}_Y) \to \mathrm{HC}_0^{\mathrm{per}}(\mathbb{A}^{\hbar}_{T^*Y}[\hbar^{-1}])$. Indeed, if $\mathcal{U} := \{U_i\}$ is a good open cover of Y, one has a natural map of complexes between the periodic cyclic-Cech complex $\mathrm{C}^{\vee}(\mathcal{U}, \mathcal{CC}^{\mathrm{per}}_{\bullet}(\mathcal{D}_Y))$ and $\mathrm{C}^{\vee}(\mathcal{V}, \mathcal{CC}^{\mathrm{per}}_{\bullet}(\mathbb{A}^{\hbar}_{T^*Y}[\hbar^{-1}]))$, where $\mathcal{V} := \{\pi^{-1}(U_i)\}$. Similarly, one has a natural map $\pi^{-1} : \mathrm{HC}_0^{\mathrm{per}}(\mathcal{O}_Y) \to \mathrm{HC}_{0,\mathbb{C}}^{\mathrm{per}}(\mathbb{A}^{\hbar}_{T^*Y})$[10]. Further, one has a trace density map $\chi_{\mathrm{FFS}} : \mathrm{HC}_0^{\mathrm{per}}(\mathbb{A}^{\hbar}_{T^*Y}[\hbar^{-1}]) \to \prod_p \mathrm{H}^{2n-2p}(T^*Y, \mathbb{C})((\hbar))$ (see [1,7,8,28]). Note that we can compose χ_{FFS} with the natural map $\beta : \mathrm{HC}_{0,\mathbb{C}}^{\mathrm{per}}(\mathbb{A}^{\hbar}_{T^*Y}) \to \mathrm{HC}_0^{\mathrm{per}}(\mathbb{A}^{\hbar}_{T^*Y}[\hbar^{-1}])$.[11] We shall abuse notation to denote $\chi_{\mathrm{FFS}} \circ \beta$ by χ_{FFS}. Let $i : Y \to T^*Y$ denote inclusion as the zero section. The following proposition is clear.

[9] Here, $\Omega^{\bullet}_{T^*X^{\mathrm{an}}}$ is the complex of sheaves on X^{an} whose sections on each open subset U of X^{an} are holomorphic forms on T^*U that are algebraic along the fibres of the projection $T^*U \to U$. d is the (holomorphic) De-Rham differential on this complex.

[10] The subscript $\mathbb{C}$ here means that the tensor product used in defining Hochschild, and hence, periodic cyclic chains is over $\mathbb{C}$.

[11] β is the composite map $\mathrm{HC}_{0,\mathbb{C}}^{\mathrm{per}}(\mathbb{A}^{\hbar}_{T^*Y}) \to \mathrm{HC}_{0,\mathbb{C}}^{\mathrm{per}}(\mathbb{A}^{\hbar}_{T^*Y}[\hbar^{-1}]) \to \mathrm{HC}_0^{\mathrm{per}}(\mathbb{A}^{\hbar}_{T^*Y}[\hbar^{-1}])$.

PROPOSITION 8. *The diagram*

$$\begin{tikzcd}
HC_0^{\mathrm{per}}(\mathcal{D}_Y) \arrow[r, "\pi^{-1}"] \arrow[d, "\chi"] & HC_0^{\mathrm{per}}(\mathbb{A}^{\hbar}_{T^*Y}[\hbar^{-1}]) \arrow[d, "\chi_{\mathrm{FFS}}"'] \\
\prod_p H^{2n-2p}(Y, \mathbb{C})((\hbar)) \arrow[r, "\pi^*"] & \prod_p H^{2n-2p}(T^*Y, \mathbb{C})((\hbar))
\end{tikzcd}$$

commutes. Further, $i^* \circ \pi^* = id$ *on* $\prod_p H^{2n-2p}(Y, \mathbb{C})((\hbar))$.

One has a "principal symbol" homomorphism $\sigma : \mathbb{A}^{\hbar}_{T^*Y} \to \mathcal{O}_{T^*Y}$. The following theorem is from [1]. The reader may also refer to [2,3] and section 7 of [31] in this context. The particular statement we want is immediate from a statement in Section 1.2.7 of [2]. The latter statement is a consequence of Theorem 4.6.1 of [2], as explained in the proof of Theorem 3.3.1 of [2].

THEOREM 4. *The following diagram commutes:*

$$\begin{tikzcd}
HC_{0,\mathbb{C}}^{\mathrm{per}}(\mathbb{A}^{\hbar}_{T^*Y}) \arrow[r, "\sigma"] \arrow[d, "\chi_{\mathrm{FFS}}"] & HC_0^{\mathrm{per}}(\mathcal{O}_{T^*Y}) \arrow[d, "I_{\mathrm{HKR}}"'] \\
\prod_p H^{2n-2p}(T^*Y, \mathbb{C})((\hbar)) & \prod_p H^{2p}(T^*Y, \mathbb{C})((\hbar)) \arrow[l, "(-)\cup\pi^* Td(T_Y)"']
\end{tikzcd}$$

The following proposition is clear as well.

PROPOSITION 9. *The following diagrams commute:*

$$\begin{tikzcd}
HC_0^{\mathrm{per}}(\mathcal{O}_Y) \arrow[r, "\pi^{-1}"] \arrow[d, "\pi^*"] & HC_{0,\mathbb{C}}^{\mathrm{per}}(\mathbb{A}^{\hbar}_{T^*Y}) \arrow[d, "id"'] \\
HC_0^{\mathrm{per}}(\mathcal{O}_{T^*Y}) & HC_{0,\mathbb{C}}^{\mathrm{per}}(\mathbb{A}^{\hbar}_{T^*Y}) \arrow[l, "\sigma"']
\end{tikzcd}$$

$$\begin{tikzcd}
HC_0^{\mathrm{per}}(\mathcal{O}_Y) \arrow[r, "\iota"] \arrow[d, "\pi^{-1}"] & HC_0^{\mathrm{per}}(\mathcal{D}_Y) \arrow[d, "\pi^{-1}"'] \\
HC_{0,\mathbb{C}}^{\mathrm{per}}(\mathbb{A}^{\hbar}_{T^*Y}) \arrow[r, "\beta"] & HC_0^{\mathrm{per}}(\mathbb{A}^{\hbar}_{T^*Y}[\hbar^{-1}])
\end{tikzcd}$$

$$\begin{tikzcd}
HC_0^{\mathrm{per}}(\mathcal{O}_Y) \arrow[r, "\pi^*"] \arrow[d, "I_{\mathrm{HKR}}"] & HC_0^{\mathrm{per}}(\mathcal{O}_{T^*Y}) \arrow[d, "I_{\mathrm{HKR}}"'] \\
\prod_p H^{2p}(Y, \mathbb{C}) \arrow[r, "\pi^*"] & \prod_p H^{2p}(T^*Y, \mathbb{C})((\hbar))
\end{tikzcd}$$

Denote the bottom arrow in the diagram of Equation (7) by θ (after extending scalars to $\mathbb{C}((\hbar))$ in the codomain). Let $\alpha \in \mathrm{HC}_0^{\mathrm{per}}(\mathcal{O}_Y)$ be arbitrary and let $\beta := I_{\mathrm{HKR}}(\alpha)$. Then,

$$
\begin{aligned}
\theta(\beta) &= \chi(\iota(\alpha)) \text{ by definition of } \theta \\
&= i^*(\pi^*(\chi(\iota(\alpha)))) \\
&= i^*(\chi_{\mathrm{FFS}}(\pi^{-1}(\iota(\alpha)))) \text{ by Proposition 8} \\
&= i^*(\chi_{\mathrm{FFS}}(\pi^{-1}(\alpha))) \text{ by Proposition 9} \\
&= i^*(I_{\mathrm{HKR}}(\sigma(\pi^{-1}(\alpha))) \cup \pi^*(\mathrm{Td}(T_X))) \text{ by Theorem 4} \\
&= i^*(I_{\mathrm{HKR}}(\sigma(\pi^{-1}(\alpha)))) \cup \mathrm{Td}(T_X) \\
&= i^*(I_{\mathrm{HKR}}(\pi^*(\alpha))) \cup \mathrm{Td}(T_X) \text{ by Proposition 9} \\
&= i^*(\pi^*(\beta)) \cup \mathrm{Td}(T_X) \text{ by Proposition 9} \\
&= \beta \cup \mathrm{Td}(T_X).
\end{aligned}
$$

Since $I_{\mathrm{HKR}} : \mathrm{HC}_0^{\mathrm{per}}(\mathcal{O}_Y) \to \prod_p \mathrm{H}^{2p}(Y, \mathbb{C})$ is an isomorphism, Proposition 3 follows from the above computation.

Acknowledgements

I am especially grateful to Xiang Tang and Boris Tsygan for very useful discussions (in particular, for helping me clarify a question I had about Theorem 4.6.1 from [2]). In fact, this note is motivated by a joint project with Xiang Tang. I am also very grateful to Damien Calaque, Giovanni Felder and Pierre Schapira for some very useful discussions. I also convey my heartfelt thanks to both the referees for their suggestions, which played an important role in improving the presentation of this paper. This work is partially supported by the Swiss National Science Foundation under the project "Topological quantum mechanics and index theorems" (Ambizione Beitrag Nr. PZ00P2_127427/1).

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